COMPARISON OF EXPERIMENTAL DATA AND SEMI-EMPIRICAL CALCULATION FOR AN INCOMPRESSIBLE TURBULENT BOUNDARY LAYER WITH PRESSURE GRADIENT

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A comparison is made of the velocity profiles obtained by semi-empirical methods of calculation and those from universal empirical relations. A method of simplifying the semiempirical calculation is proposed.

Great success has recently been attained in the area of purely empirical calculation of the turbulent boundary layer. Statistical treatment of the experimental data has given a generalization of velocity profiles in the boundary layer of equilibrium flows [3], and universal velocity profile shapes have been obtained in the inner and outer parts of the layer. There have been numerous attempts [4–7] to use these ideas to construct a method of calculating the integral characteristics of the boundary layer.

It is therefore interesting to examine how far the results of the semi-empirical theory agree with universal empirical relations.

The semi-empirical methods assume a two-layer form of boundary layer. It is assumed that in the immediate vicinity of the wall, where turbulent fluctuations are damped, there is a laminar sublayer in which viscous friction has the greatest influence. The velocity profile in the laminar sublayer is determined by the Pohlhausen [1] method, and its thickness is found by dimensional analysis to be $\eta_l = \delta_l/\delta = \alpha/R_1$. The coefficient α is regarded as being a universal constant, equal to about 11.5.

In the turbulent part of the layer the connection between the shear stress τ and the transverse velocity gradient is determined by the Prandtl "mixing length" formula $\tau = \rho l^2 (\partial u/\partial y)^2$. If the dependence of τ and l on y is known, we may obtain the velocity profile in the turbulent part of the layer. The constant of integration may then be found from the condition that the velocity profile is continuous at the edge of the laminar sublayer or at the outer edge of the boundary layer. In the first case the velocity profile may be expressed in a form reminiscent of the law of the wall

$$\frac{u}{u_*} = \alpha + \frac{\alpha^2}{2}\frac{A}{R_1} + \int\limits_{\eta_l}^{\eta} \frac{\delta}{l}\sqrt{\frac{\tau}{\tau_0}}\, d\eta, \qquad (1)$$

and in the second case—the velocity defect law

$$\frac{u - u_\delta}{u_*} = \int\limits_{1}^{\eta} \frac{\delta}{l}\sqrt{\frac{\tau}{\tau_0}}\, d\eta. \qquad (2)$$

If the constant of integration is determined from one of the conditions mentioned above, then fulfillment of the second condition is only possible when there is a definite relation between the shear stress at the wall and the boundary layer thickness, the so-called resistance law,

$$\frac{u_\delta}{u_*} = \alpha + \frac{\alpha^2}{2}\frac{A}{R_1} + \int\limits_{\alpha/R_1}^{1} \frac{\delta}{l}\sqrt{\frac{\tau}{\tau_0}}\, d\eta.$$

In order to determine the specific form of the velocity profile and the resistance law, we must assign the integrand functions in the foregoing formulas.

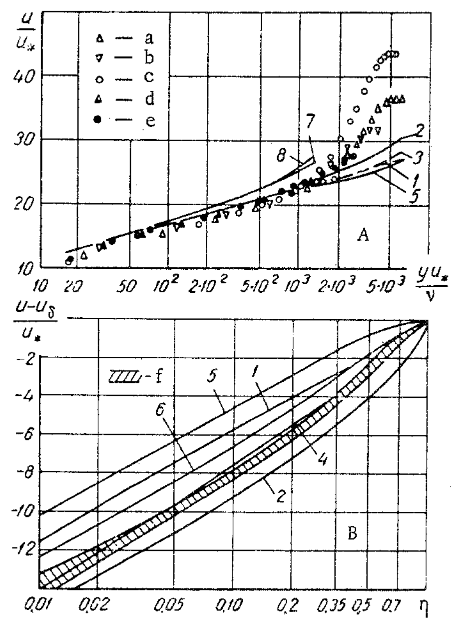

Fig. 1. Comparison of calculated (A—from (1), B—from (2)) and experimental (A—from the data of [3], B—from [4]) velocity profiles in the boundary layer on a flat plate: 1, 3, 5, 7) with $l = 0.4y$; 2, 4, 6, 8) $l/\delta = 0.14 - 0.08 \times (1 - \eta)^2 - 0.06(1 - \eta)^4$, 1, 2) with $\tau/\tau_0 = 1$; 3, 4) $\tau/\tau_0 = 1 - \eta^2$; 5, 6) $\sqrt{\tau/\tau_0} = 1 - \eta^2$; 7) $\sqrt{\tau/\tau_0} = 1 + 0.5A\eta - (1 + 0.5A)\eta^2$, $A = 20$; 8—$\tau/\tau_0 = 1 + A\eta - (1 + A)\eta^2$, $A = 20$; a) $A/R_1 = 2.84 \cdot 10^{-3}$; b) $2.59 \cdot 10^{-3}$; c) $5.15 \cdot 10^{-3}$; d) $A/R_1 = -0.20 \cdot 10^{-3}$; e) $A = 0$; f) experiment.

From Nikuradze's tests Prandtl obtained the empirical dependence of "mixing length" on the dimensionless transverse coordinate $l/\delta = 0.14 - 0.08 \times (1 - \eta)^2 - 0.06(1 - \eta)^4$, which in most cases is replaced, to simplify calculation, by the simpler formula $l/\delta = 0.4\eta$.

Following the suggestion of Fedyaevskii [1], τ/τ_0 or $\sqrt{\tau/\tau_0}$ is replaced by polynomials in which the coefficients, which are functions of the parameter A, are determined from the boundary conditions at the wall and at the outer edge of the boundary layer.

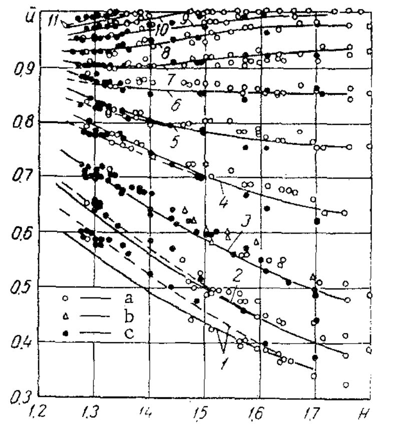

Fig. 2. Comparison of calculated and experimental velocity profiles in a boundary layer with pressure gradient: a) results of Doenhoff and Tetervin at Re = $(0.9-4.18)\cdot 10^6$; b) results of Brebner; c) results of Schubauer and Klebanoff at Re = $27.7\cdot 10^6$; full lines—calculation at lg R** = 3.4; dotted lines at lg R** = 6.2; 1) with $y/\delta^{**} = 1/4$; 2) 1/2; 3) 1; 4) 2; 5) 3; 6) 4; 7) 5; 8) 6; 9) 7; 10) 8; 11) 9.

Depending on the number of conditions used, we may obtain polynomials of various degrees, with various accuracies of approximation to the shear stress distribution.

Recently, other, stricter methods of finding the shear stress have been proposed and may be used when required for improved calculations. However, as indicated above, the characteristic properties of velocity profiles in the boundary layer may be revealed by semi-empirical methods, even for quite a rough approximation of the shear stress.

This may be followed most easily in the example of the boundary layer on a flat plate at constant pressure. Figure 1 A compares experimental velocity profiles near the wall, taken from [4], with calculations according to (1), using various approximating relations to determine the "mixing length" and the shear stress profile. Also given are curves for a boundary layer with positive pressure gradient (A = 20). It is clear that lack of accuracy in the approximation formulas for shear stress and "mixing length" does not appreciably affect the velocity distribution in the inner part of the boundary layer. In all cases the calculated values prove to be extremely close to the logarithmic velocity profile. The positive pressure gradient in the external stream has practically no influence on the velocity profile in the inner part of the boundary layer.

Thus, the comparison indicates that results obtained from semi-empirical methods are in good agreement with the empirical law of the wall.

Figure 1,B compares experimental data for a flat plate boundary layer [3] with calculated values from (2), using various dependences for τ and l. The results show that the accuracy of approximation to the velocity in the outer part of the boundary layer depends appreciably on correct choice of the functions which determine the shear stress profile and the "mixing length." The better these functions reflect the actual distribution of τ and l, the closer are the calculated results to the actual velocity profile. The best agreement between calculated and experimental characteristics is observed when we use a polynomial for τ/τ_0 and the first formula for l/δ, and the worst agreement is obtained when we use a second-degree polynomial for $\sqrt{\tau/\tau_0}$ and the simplified formula for l/δ.

Values of l/δ found from the simplified formula are appreciably greater than the actual values in the outer part of the boundary layer, causing the calculated velocity profiles to be fuller than the actual ones, and leading to reduction of the parameters $H = \delta^*/\delta^{**}$. By treating the curves shown in Fig. 1B, we may obtain the formula $H = (1 - \beta\sqrt{c_f/2})^{-1}$, which relates H to the local friction coefficient c_f. Depending on the velocity profile fullness the coefficient β in this formula will vary from 4.7 (curve 5) to 6.8 (curve 4); in the latter case the coefficient practically coincides with the empirical value given by Hama [10].

It is clear from this that the cause of inaccuracy in H for some of the semi-empirical methods is the use of the simplified formula to determine the "mixing length." Using the more accurate formulas for τ/τ_0 and l/δ, values of H closer to experiment may be obtained.

The closeness of approximation to τ/τ_0 and l/δ has very little influence on the resistance law. Thus, using the simplified formula for l/δ, and for $\sqrt{\tau/\tau_0}$ a second-degree polynomial [2] (the corresponding velocity profile is shown in Fig. 1,B, curve 5), we find that the resistance law for a plate may be represented, over a wide range of Reynolds numbers ($R^{**} = 10^3-10^7$), by the power relation

$$c_{f0}/2 = 0.00652\ R^{**^{-0.16}}, \tag{3}$$

which practically coincides with the analogous formula of Faulkner [10], obtained by processing a large number of experimental data.

We shall pass on to examine profiles for boundary layers with longitudinal pressure gradient.

By reducing a large number of measurements on straight airfoils, Doenhoff and Tetervin [7], and later Schubauer and Klebanoff [8], obtained a one-parameter family of universal velocity profiles $\bar{u} = \bar{u}(H, y/\delta^{**})$ for boundary layers with longitudinal pressure gradient.

Figure 2 compares the theoretical and experimental dependences of $\bar{u} = \bar{u}(H)$ at y/δ^{**} = const. In calculating the velocities it was assumed that τ/τ_0 is determined

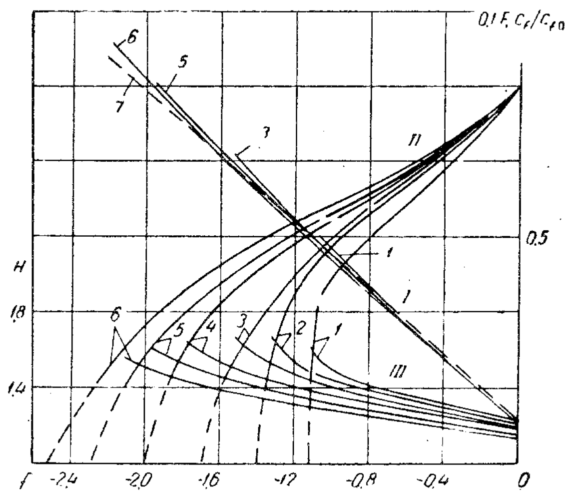

Fig. 3. Dependence of local boundary layer characteristics (I—F; II—c_f/c_{f_0}; III—H) on shape factor f, calculated on the basis of Fedyaevskii's method: 1) with lg R** = 3.4; 2) 3.8; 3) 4.2; 4) 4.6; 5) 5.4; 6) 6.2; 7) with F = 1.17–4.4f.

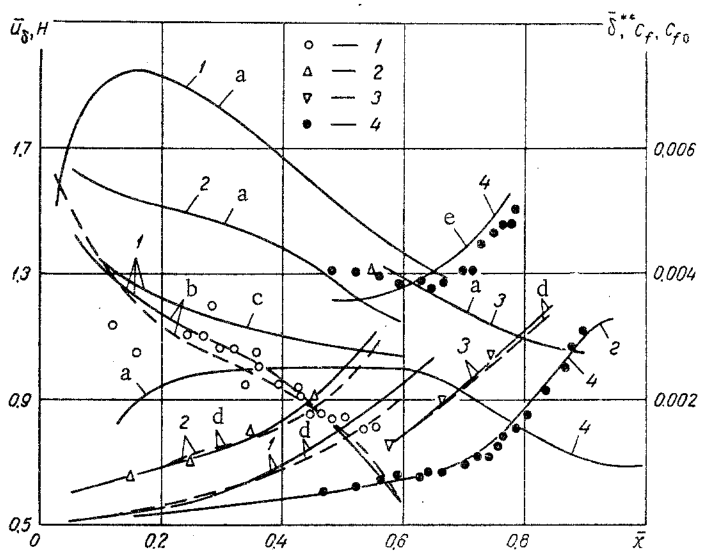

Fig. 4. Comparison of calculated characteristics of a plane boundary layer with the experimental data of different authors (a) u_δ; b) c_{f_0}; d) δ^{**}; e) H): 1) Fomina and Buchinka (Re = $5.17 \cdot 10^6$); 2) Doenhoff and Tetervin (Re = $2.64 \cdot 10^6$); 3) Grushvitets (Re = $0.85 \cdot 10^6$); 4) Schubauer and Klebanoff (Re = $27.7 \cdot 10^6$); dotted lines—calculation according to [2]; solid lines-according to (5).

[2] by a second-degree polynomial (using two conditions at the wall and one at the outer edge of the layer), and the "mixing length" by the simplified formula $l = 0.4y$.

Over the whole range of H, right up to unseparated conditions of the boundary layer (H = 1.75–1.8), the calculated results are in good agreement with experimental data. Comparison of the calculated curves shows that the influence of Reynolds number variation on the above type of relation is small. Some systematic divergence of the experimental results at different Reynolds numbers is observed only at small distance from the wall, and is quite well described by theory. It should be kept in mind, however, that the calculated values of H in the case examined differ somewhat from the experimental values. Therefore for the same value of H the theoretical and experimental velocity profiles correspond to different values of shape factor A. It has been shown that the lack of agreement may be reduced by use of more accurate approximations for the shear stress profile and the "mixing length."

Finally, we shall examine one possible means of simplifying the semiempirical calculation of boundary layer characteristics in the presence of longitudinal pressure gradient. We shall introduce the shape factor $f = (2\delta^{**}/c_{f})(\bar{u}_\delta'/\bar{u}_\delta)$, where δ^{**} is the momentum thickness, c_{f_0} is the local friction coefficient in zero-gradient flow, corresponding to the actual c_{f_0}. Expressing the momentum thickness in terms of the shape factor f with the aid of the formula given above, we may transform the integral momentum relation for the axisymmetric boundary layer, in the case where the layer thickness is small compared with the radius r_0 of transverse curvature of the body, to the following form:

$$\frac{df}{dx} = \frac{u_\delta'}{u_\delta} F(f, R^{**}) + \left[\frac{u_\delta''}{u_\delta'} - (m+1)\frac{r_0'}{r_0}\right] f,$$

$$F(f, R^{**}) = (m+1)c_f/c_{f0} - $$

$$- [2 + (m+1)(1+H)] f,$$

$$m = -d\lg(c_f/2)/d\lg R^{**}. \quad (4)$$

The function $F(f, R^{**})$ was calculated using Fedyaevskii's method [2]. It was assumed that the shear stress profile is determined by a second-degree polynomial for $\sqrt{\tau/\tau_0}$, and the "mixing length" by the simplified formula $l = 0.4\,y$. In the entire range of change of shape factor f the curves of function $F(f, R^{**})$ constructed for different values of R^{**} do not deviate from the straight line $F = 1.17 - 4.4f$ by more than 2–3%. Only near the points $f = f_S$, corresponding to $c_f = 0$, is a greater divergence, reaching 5%, observed between the values of function F and the linear relation. It should be noted that the coefficients in the last equality are close to the corresponding coefficients in Loitsyanskii's method [10]. By replacing F in (4) by its linear approximation and performing the integration, we obtain a solution, using (3), of the integral momentum relation in the following form:

$$R^{**} = \frac{0.0132}{\bar{r}_0^{1.88}\,\bar{u}_\delta^{2.07}}\left[153\,\bar{u}_{\delta t}^{2.4}\,\bar{r}_{0t}^{2.18}\,R_t^{**\,1.16} + 1.17\,\mathrm{Re}\int_{\bar{x}_t}^{\bar{x}} \bar{u}_\delta^{3.4}\,\bar{r}_0^{2.18}\,d\bar{x}\right]^{0.863}, \quad (5)$$

where the subscript t denotes values of the characteristics at the point x_t from which calculation of the turbulent boundary layer is begun (for instance, at the transition point). Putting $\bar{r}_0 = 1$ in (5), we obtain a formula for calculating the characteristics of a plane boundary layer.

The comparison that has been made for a large number of different problems has shown (Fig. 4) that the solution (5) is in good agreement with experimental data and differs extremely little from the results obtained on the basis of the initial method of calculation [2].

Thus, the data presented show that the conclusions which follow from semiempirical relations for velocity profiles, but the methods themselves may be considerably simplified on the computational side, and made as operational as the simplest empirical methods. Moreover, semiempirical methods possess considerably greater flexibility. They are generalized in the case of gas flow with large velocities [11] in the presence of heat transfer, permit adjustment for the influence of transverse body curvature [12], the presence of chemical reactions, etc. Therefore, further development and improvement of these methods is a matter of great interest.

NOTATION

x, y—coordinate axes along and perpendicular to surface; u—projection of velocity on x axis; r_0—radius of transverse body curvature; $\bar{u}_\delta = u_\delta/V$; $\bar{x} = x/V$; $\bar{r}_0 = r_0/b$; Re = Vb/ν; V and b—characteristic velocity and linear dimension; ν—kinematic viscosity; $u_* = \sqrt{\tau_0/\rho}$—"dynamic" velocity; τ_0—shear stress at wall; $c_f = 2\tau_0/\rho u_\delta^2$—local friction coefficient; ρ—air density; δ—boundary layer thickness; δ^*—displacement thickness; δ^{**}—momentum thickness; $H = \delta^*/\delta^{**}$; $R^{**} = u_\delta\delta^{**}/\nu$; $\eta = y/\delta$; $R_1 = u_*\delta/\nu$; $A = (\delta/\tau_0)dp/dx$; l—"mixing length."

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